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LETTER TO THE EDITOR

Scaling behaviour in extended coalescing random walker model

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Abstract. A simple aggregation model on one-dimensional lattice is presented to study the scaling behaviour. The model is an extended version of the coalescing random walker model to take into account the dependence of transition probability upon mass s of particle. A particle moves ahead one step with transition probability T and is stopped with probability $1-T$ where $T=a+bs^{-\alpha}$ ($\alpha>0$, $a,b\geqslant 0$ and $a+b\leqslant 1$). It is shown that the mean mass $\langle s\rangle$ of particle scales as $\langle s\rangle\approx t^{\beta}$ where t is time. The scaling relation $\beta=1/(1+\alpha)$ is satisfied for $a=0.0$. For $a>0$, the scaling relation $\beta=\max[0.5, 1/(1+\alpha)]$ is satisfied. We discuss the relation between our model and the extended KPZ equation.

Recently, there has been increasing interest in the scaling structures of growth processes such as the cluster–cluster aggregation (CCA) model, the rough surface model, the diffusion-limited aggregation (DLA) model, and the river network model [1–3]. Considerable work has already been performed on the scaling properties of non-equilibrium fractal growth. Most attempts to develop a theoretical understanding of fractal growth focused directly or indirectly on the screening process. Even for very simple models such as DLA this has proven to be a formidable challenge. For the case of a growing surface, some analytical attempts have succeeded in deriving the scaling exponents. The main approach for describing the growth of surfaces and interfaces is based on coarse-grained Langevin-type equation [3]. Kardar, Parisi and Zhang (KPZ) [4] have presented a nonlinear interface equation. In two dimensions, the scaling of the interface width $w(L,t)$ on length scale L at time t for the KPZ equation have been shown from a renormalization-group analysis to be $w(L,t)=L^{1/2}f(t/L^{3/2})$ where $f(x)\approx x^{1/3}$ for $x\ll 1$ and $f(x)\approx$ constant for $x\gg 1$[].

The scaling exponent of the cluster-size distribution in a simple aggregation model with injection has analytically been derived by Takayasu [5]. However, the governing equation such as the KPZ equation has been unknown until now for aggregation models. The scaling exponents for aggregation models have rarely been derived from analytical method.

In this letter, we present a simple aggregation model on one-dimensional lattice to be described by the diffusion equation with generalized nonlinearity. We study the scaling behaviour of the aggregation process. In order to obtain the lattice model for the nonlinear diffusion equation, we extend the coalescing random walker model to take into account the dependence of transition probability upon mass s of a particle. When two particles collide with each other, they coalesce. Each cluster (or particle) is infinitesimal but has a finite mass. We show that the mean mass $\langle s\rangle$ of a particle scales as $\langle s\rangle\approx t^{\beta}$, the scaling relation $\beta=1/(1+\alpha)$ is satisfied for $a=0.0$, and for $a>0$, the scaling relation $\beta=\max[0.5, 1/(1+\alpha)]$ is satisfied. In the limit of $b=0$ and $a=1/2$, this model reduces

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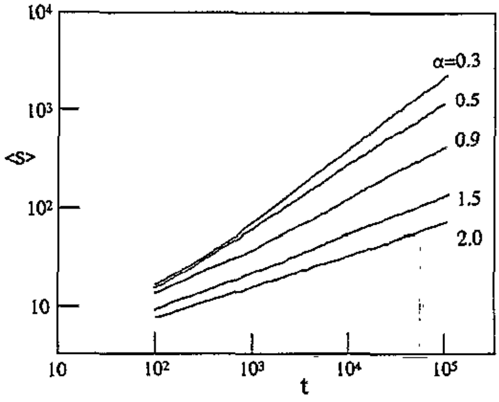

Figure 1. The log–log plot of the mean mass $\langle s\rangle$ against time t for $\alpha = 0.3, 0.5, 0.9, 1.5$ and 2.0 in the case of $a = 0$.

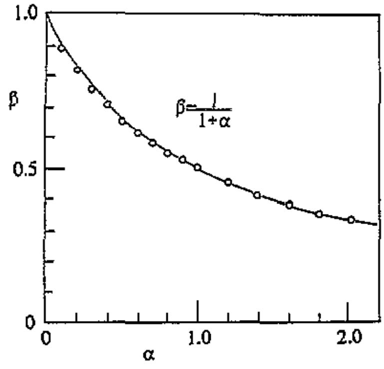

Figure 2. The plot of the scaling exponent β against α in the case of $a = 0$. The solid curve represents the relation $\beta = 1/(1+\alpha)$.

to the well known irreversible one-species coagulation model [6]. This model is closely related to the diffusion-limited reaction [7]. Also, the aggregation model with injection in the limit is consistent with the Scheidegger's river-network model [8].

Our aggregation model is defined on a one-dimensional lattice of L sites with periodic boundary conditions. Each site is occupied by one particle or it is empty. We extend the coalescing random walker model to take into account the mass-dependent transition probability T. In our model, transition probability T depends only on the cluster mass s. For an arbitrary configuration, one update of the system is performed in parallel for all particles. A particle moves ahead one step with transition probability $T = a + bs^{-\alpha}$ and is stopped with probability $1 - T$. When two particles with masses s_1 and s_2 collide with each other, they coalesce and the resultant particle has the mass $s_1 + s_2$. Each cluster (or particle) is infinitesimal but a finite mass. Let $s(i, t)$ be the mass of the particle on the site i at the t time step. The aggregation can be represented by the stochastic equation for $s(i, t)$

$$s(i, t+1) = [1 - w(i, t)]s(i, t) + w(i-1, t)s(i-1, t) \tag{1}$$

where $w(i, t)$ is a stochastic variable which is equal to 1 with mass-dependent probability $T = a + bs^{-\alpha}$ ($a > 0$, $a, b \geqslant 0$ and $a + b \leqslant 1$) when the particle on the ith site jumps to the $(i+1)$th site and which is equal to 0 with probability $1 - T$ when the particle on the ith site does not jump to the $(i+1)$th site. In the limit of $a = 1/2$ and $b = 0$, the aggregation is consistent with the ordinary coalescing random walker model. We restrict ourselves to the case of $\alpha > 0$ since the mean cluster size $\langle s\rangle$ does not scale for the case of $\alpha < 0$. The aggregation process with $\alpha < 0$ cannot be simulated by our aggregation model.

We perform simulations of our aggregation model according to (1) for the system size $L = 10^5$. Each run is calculated until 10^5 time steps. We study the scaling behaviour of the cluster-mass distribution. We define the mean mass $\langle s\rangle$ of particles as

$$\langle s\rangle \equiv \sum_{s=1}^{\infty} s^2 h_s \Big/ \sum_{s=1}^{\infty} sn_s \tag{2}$$

where n_s is the cluster-mass distribution with mass s. Figure 1 shows the log–log plot of the mean mass $\langle s\rangle$ against time t for $\alpha = 0.3, 0.5, 0.9, 1.5$ and 2.0 where $a = 0$. The mean mass $\langle s\rangle$ scales as

$$\langle s\rangle \approx t^{\beta}. \tag{3}$$

Figure 2 shows the plot of the scaling exponent β against α. The solid curve represents the relation

$$\beta = 1/(1+\alpha). \tag{4}$$

The data agree with the scaling relation (4).

The scaling relation (4) is derived from a simple scaling argument as follows. The increment of the typical mass by the coalescence per unit time is proportional to the transition probability. Then, the following equation is satisfied

$$\mathrm{d}s/\mathrm{d}t \approx s^{-\alpha}. \tag{5}$$

The particle mass s scales as $s \approx t^{1/(1+\alpha)}$. We calculate the case of $a = b = 1/2$. Figure 3 shows the log–log plot of the mean mass $\langle s \rangle$ against time t for $\alpha = 0.3, 0.5, 1.0, 1.5$ and 2.0. Figure 4 shows the plot of the scaling exponent β against α. For $0 < \alpha < 1$, the scaling exponent β is given by (4). For $\alpha \geqslant 1$, the exponent β becomes the constant value 0.5. The scaling exponent β is represented by

$$\beta = \max[1/2, 1/(1+\alpha)]. \tag{6}$$

The value $\beta = 1/2$ of the scaling exponent β agrees with that obtained by the coalescing random walker model. When the power α is larger than 1, the aggregation process is governed by the ordinary coalescing random walk.

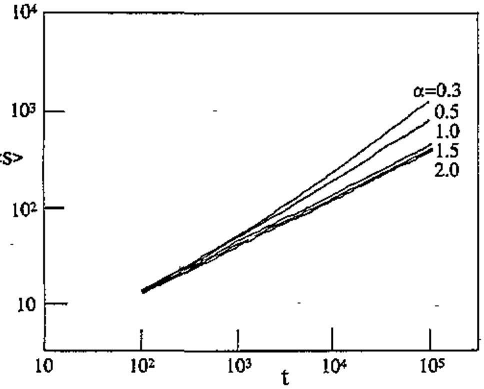

Figure 3. The log–log plot of the mean mass $\langle s \rangle$ against time t for $\alpha = 0.3, 0.5, 1.0, 1.5$ and 2.0 in the case of $a = b = 1/2$.

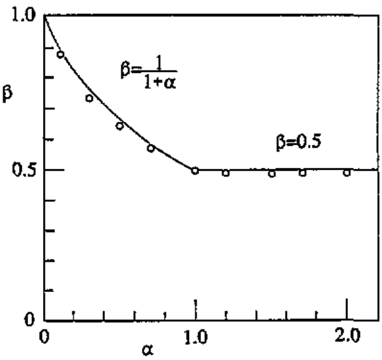

Figure 4. The plot of the scaling exponent β against α in the case of $a = b = 1/2$. The solid curve represents the relation $\beta = \max[1/2, 1/(1+\alpha)]$.

We calculate the cluster-mass distribution n_s. The cumulative mass distribution N_s is defined as $N_s = \sum_{s'=s}^{\infty} n_{s'}$. Figure 5 shows the semi-log plot of the cumulative mass distribution N_s against the mass s for $t = 10^3, 3 \times 10^3, 10^4$ and 3×10^4 where $\alpha = 0.5$. In order to study the scaling form of the cumulative mass distribution, we plot the rescaled cumulative distribution against the rescaled mass. Figure 6 shows the semi-log plot of the rescaled cumulative distribution $t^{0.652} N_s$ against the rescaled mass $t^{-0.652} s$ for the data in figure 5. The data collapses on a curve. We find that the cumulative mass distribution is described in terms of

$$N_s \approx \langle s \rangle^{-1} f(s/\langle s \rangle) \tag{7}$$

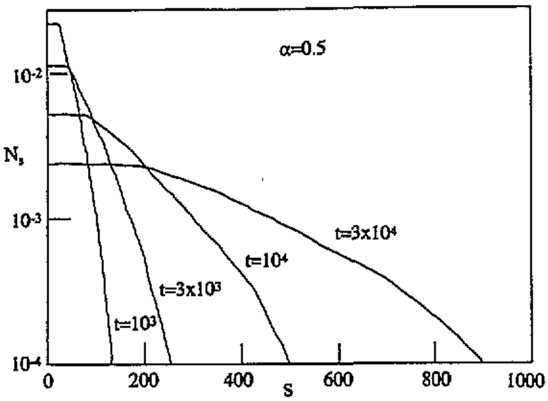

Figure 5. The semi-log plot of the cumulative mass distribution N_s against mass s for $t = 10^3, 3 \times 10^3, 10^4$ and 3×10^4 where $\alpha = 0.5$.

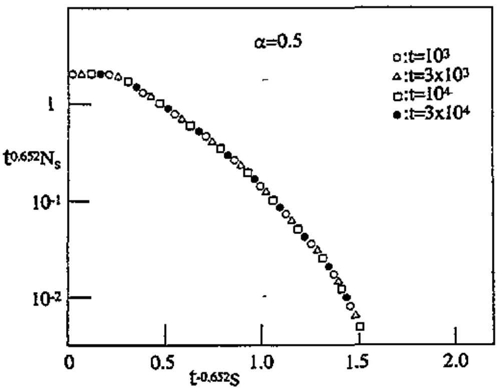

Figure 6. The semi-log plot of the rescaled cumulative distribution $t^{0.652}N_s$ against the rescaled mass $t^{-0.652}s$ for the data in figure 5.

where the scaling function $f(x)$ is nearly a Gaussian distribution and $\langle s \rangle \approx t^{\beta}$.

We consider the governing equation for the aggregation process. We discuss the relation between our result and the KPZ equation with generalized nonlinearity. The number of particles decreases with time by coalescence. The typical mass $\langle s \rangle$ of particles increases with time. The particle number is proportion to the inverse $\langle s \rangle^{-1}$ of the typical mass on the one-dimensional system. The density ρ is proportion to $\langle s \rangle^{-1}$. In the limit of $a = 1/2$ and $b = 0$, the aggregation process reduces to the ordinary coalescing random walker model. Then, the typical mass $\langle s \rangle$ increases as $\langle s \rangle \approx t^{1/2}$ due to the Brownian motion. The density ρ decreases as $\rho \approx t^{1/2}$. The density ρ is governed by the ordinary diffusion equation. In our case, at a coarse-grained scale, the density changes by diffusion and convection. The convection term is represented by $\nabla \rho^{1+\alpha}$ since the drift velocity of particles is proportional to the transition rate $s^{-\alpha}$. Then, the density ρ is governed by the extended Burgers equation with generalized nonlinearity:

$$\partial \rho / \partial t = \upsilon \nabla^2 \rho + \lambda \nabla \rho^{1+\alpha}. \tag{8}$$

The extended Burgers equation is transformed to the generalized KPZ equation

$$\partial h/\partial t = v\nabla^2 h + \lambda|\nabla h|^{1+\alpha} \qquad (9)$$

where $\rho = -\nabla h$. Krug and Spohn [9] have found that the dynamic exponent z is given by

$$z = \min[2, -\zeta\alpha + \alpha + 1] \qquad (10)$$

where ζ is the static exponent. The dynamic exponent z equals our scaling exponent β^{-1} since the characteristic time t scales as $t \approx 1^{Z}$. The static exponent ζ represents the exponent of the covariance $\langle (h(x,t) - h(x',t))^2 \rangle \approx |x' = x|^{2\zeta}$. In our case of the random initial configuration, $\zeta = 0$. The scaling relation (6) is obtained. In the limit of $a = 0$, $v = 0$ and the scaling relation (4) is obtained.

In summary, we presented a simple aggregation process described by the nonlinear diffusion equation. We calculated the scaling exponent by computer simulation. We found that the scaling exponent agrees with that derived from the nonlinear diffusion equation.

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